

ROKHLIN DIMENSION FOR COMPACT GROUP ACTIONS

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ABSTRACT. We introduce and systematically study the notion of Rokhlin dimension (with and without commuting towers) for compact group actions on C^* -algebras. This notion generalizes the one introduced by Hirshberg, Winter and Zacharias for finite groups, and contains the Rokhlin property as the zero dimensional case. We show, by means of an example, that commuting towers cannot always be arranged, even in the absence of K -theoretic obstructions. For a compact Lie group action on a compact Hausdorff space, freeness is equivalent to finite Rokhlin dimension of the induced action. We compare the notion of finite Rokhlin dimension to other existing definitions of noncommutative freeness for compact group actions. We obtain further K -theoretic obstructions to having an action of a non-finite compact Lie group with finite Rokhlin dimension with commuting towers, and use them to confirm a conjecture of Phillips.

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1. INTRODUCTION

The study of group actions on C^* -algebras, as well as their associated crossed products, has been the object of very intensive research since the early beginnings of the operator algebra theory. Both in the von Neumann algebra and in the C^* -algebra case, crossed products have provided a great deal of highly nontrivial examples via a construction that combines the dynamical properties of the action together with the structural properties of the underlying algebra.

A crucial result in the context of measurable dynamics is the Rokhlin Lemma,

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which asserts that an aperiodic measure preserving action of \mathbb{Z} can be “approximated”, in a suitable sense, by finite cyclic shifts. Its reformulation in terms of outer automorphisms of (commutative) von Neumann algebras using partitions of unity consisting of orthogonal projections has led to a number of versions of the Rokhlin property, both in the von Neumann algebra and in the C^* -algebra case. See [23] and [18] for integer actions, and see [12], [16], [17] and [26] for the finite group case.

On the side of topological dynamics, the notion of freeness of a group action is central. Recall that an action of a group G on a space X is said to be *free* if no non-trivial group element of G acts with fixed points. Viewing C^* -algebras as noncommutative topological spaces, it is natural to look for generalizations of the concept of freeness to the case of group actions on C^* -algebras. It turns out that there is not a single version of noncommutative freeness. The book [28] provides a detailed presentation and comparison of a number of them, mainly for compact Lie groups, including (locally) discrete K -theory, (total) K -freeness, (hereditary) saturation, and others. The more recent survey article [29], which considers mostly finite groups, incorporates other notions that have been intensively studied in the past two decades: the Rokhlin property and the tracial Rokhlin property. We refer the reader to the introduction of [29] for a motivation of the study of free actions on C^* -algebras.

More recently, Hirshberg, Winter and Zacharias introduced in [15] the notion of finite Rokhlin dimension for finite group actions (as well as automorphisms), as a generalization of the Rokhlin property. This more general notion has the Rokhlin property as its zero dimensional case, and moreover has the advantage of not requiring the existence of projections in the underlying algebra. Finite Rokhlin dimension is in particular much more common than the Rokhlin property.

The paper [13] consists of a further study of finite Rokhlin dimension, where the authors extend the notion to the non-unital setting, and also derive some K -theoretical obstructions in the commuting tower version. These obstructions are used to show that no non-trivial finite group acts with finite Rokhlin dimension with commuting towers on either the Jiang-Su algebra \mathcal{Z} , or the Cuntz algebra \mathcal{O}_∞ . There, Hirshberg and Phillips introduce the notion of X -Rokhlin property for an action of a compact Lie group G and a compact free G -space X . They show that when G is finite and X is finite dimensional, the X -Rokhlin property is equivalent to having finite Rokhlin dimension with commuting towers in the sense of Hirshberg-Winter-Zacharias.

Our work develops the concept of Rokhlin dimension for compact group actions on unital C^* -algebras, generalizing the case of finite group actions of [15], the Rokhlin property in the compact group case as in [14], and including the X -Rokhlin property from [13], which is shown to be equivalent to finite Rokhlin dimension with commuting towers in our sense, at least for Lie groups. The starting point of this project was the simple observation that if $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$ is an action of the circle with the Rokhlin property on a unital C^* -algebra A , and if n is a positive integer, then the restriction of α to the finite subgroup $\mathbb{Z}_n \subseteq \mathbb{T}$ has Rokhlin dimension at most one. Theorem 3.10 can be regarded as a significant generalization of this fact.

This paper is organized as follows. In Section 2, we establish the notation that will be used throughout the paper, and briefly recall the necessary preliminary

definitions and results. In Section 3, we introduce and systematically study the notion of Rokhlin dimension for compact group actions on unital C^* -algebras. In particular, we show that finite Rokhlin dimension is preserved under a number of constructions, namely tensor products, direct limits, passage to quotients by invariant ideals, and restriction to closed subgroups of finite codimension. In Section 4, we compare the notion of having finite Rokhlin dimension (mostly in the commuting tower case) with other existing forms of freeness of group actions on C^* -algebras. We show that in the commutative case, finite Rokhlin dimension is equivalent to freeness of the action on the maximal ideal space; see Theorem 4.2. Moreover, for a compact Lie group action, the formulation with commuting towers is equivalent to the X -Rokhlin property introduced in [13]; see Theorem 4.5. We apply this to deduce that actions with finite Rokhlin dimension with commuting towers have discrete K -theory and are totally K -free. Theorem 4.16 establishes K -theoretic obstructions that are complementary to the ones established in [13]. We use this result to confirm a Conjecture of Phillips from [28]; see Remark 4.17. Our results in fact show that Phillips' conjecture holds for a class of C^* -algebras which is much larger than the class of AF-algebras, without assuming that the action is specified by the way it is constructed.

We show in Example 4.8 that commuting towers cannot always be arranged, even at the cost of considering additional towers, and even in the absence of K -theoretic obstructions. Indeed, this example (originally constructed by Izumi in [16]) has Rokhlin dimension 1 with noncommuting towers, and infinite Rokhlin dimension with commuting towers. Theorem 4.21 collects and summarizes the known implications between the notions considered in Section 4, and it also references counterexamples that show that no other implications hold in full generality. Finally, in Section 5, we give some indication of possible directions for future work, and raise some natural questions related to our findings.

Some applications of the results in this paper will appear in [8] and [10].

All groups will be second countable. (Many statements are true in greater generality, but second-countable groups suffice for our purposes.) By a theorem of Birkoff-Kakutani (Theorem 1.22 in [24]), a topological group is metrizable if and only if it is first countable. In particular, all our groups will be metrizable. It is easy to check that a compact metrizable group admits a translation-invariant metric. We will implicitly choose such a metric on all our groups, which will usually be denoted by d .

We point out that most of our results are stated and proved for arbitrary second-countable compact groups, while some results in Section 4 are only obtained for compact Lie groups.

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discussions concerning the existence of local cross sections for free actions. He is especially indebted to Ilan Hirshberg and Chris Phillips for giving him access to a draft of their paper [13] long before it was ready for publication, and for helpful discussions and correspondence. Finally, he would also like to thank Wilhelm Winter for suggesting this line of research.

2. NOTATION AND PRELIMINARIES

We adopt the convention that $\{0\}$ is not a unital C^* -algebra, this is, we require that $1 \neq 0$ in a unital C^* -algebra. Homomorphisms of C^* -algebras are always assumed to be $*$ -homomorphisms. For a C^* -algebra A , we denote by $\text{Aut}(A)$ the automorphism group of A . If A is moreover unital, then $\mathcal{U}(A)$ denotes the unitary group of A .

For a locally compact group G , an action of G on A is always assumed to be a *continuous* group homomorphism from G into $\text{Aut}(A)$, unless otherwise stated. If $\alpha: G \rightarrow \text{Aut}(A)$ is an action of G on A , then we will denote by A^α the fixed-point subalgebra of A under α .

We take $\mathbb{N} = \{1, 2, \dots\}$. The p -adic integers will not appear in this paper, so we write \mathbb{Z}_n for the cyclic group $\mathbb{Z}/n\mathbb{Z}$. If A is a C^* -algebra, we denote by $[\cdot, \cdot]: A \times A \rightarrow A$ the additive commutator, this is, $[a, b] = ab - ba$ for all $a, b \in A$.

2.1. Equivariant K -theory. We recall the definition of equivariant K -theory for compact group actions. A thorough development of the theory can be found in [28], whose notation we will follow. We denote the suspension of a C^* -algebra A by SA .

Definition 2.1. Let G be a compact group, let A be a unital C^* -algebra and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action. Denote by $\mathcal{P}_G(A)$ the set of all G -invariant projections in all of the algebras of the form $\mathcal{B}(V) \otimes A$, for all finite dimensional representations $\lambda: G \rightarrow \mathcal{U}(V)$ (we take the diagonal action of G on $\mathcal{B}(V) \otimes A$). Two G -invariant projections p and q in $\mathcal{P}_G(A)$ are said to be *equivariantly Murray-von Neumann equivalent* if there exists a G -invariant partial isometry s in $\mathcal{B}(V, W) \otimes A$ such that $s^*s = p$ and $ss^* = q$. Given a finite dimensional representation $\lambda: G \rightarrow \mathcal{U}(V)$ of G and a G -invariant projection $p \in \mathcal{B}(V) \otimes A$, and to emphasize role played by the representation λ , we denote the element in $\mathcal{P}_G(A)$ it determines by (p, V, λ) . We let $S_G(A)$ denote the set of equivalence classes in $\mathcal{P}_G(A)$ with addition given by direct sum.

We define the *equivariant K_0 -group* of A , denoted $K_0^G(A)$, to be the Grothendieck group of $S_G(A)$, and define the *equivariant K_1 -group* of A , denoted $K_1^G(A)$, to be $K_0^G(SA)$, where the action of G on SA is trivial in the suspension direction.

Remark 2.2. The equivariant K -theory of A is a module over the representation ring $R(G)$ of G , which can be identified with $K_0^G(\mathbb{C})$, with the operation given by tensor product. The induced operation $R(G) \times K_*^G(A) \rightarrow K_*^G(A)$ makes $K_*^G(A)$ into an $R(G)$ -module.

The following result is Julg's Theorem.

Theorem 2.3. (See Theorem 2.6.1 in [28]) Let G be a compact group, let A be a unital C^* -algebra and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action. Then there is a natural isomorphism

$$K_*^G(A) \cong K_*(A \rtimes_\alpha G).$$

When the group G is abelian, we denote its dual group by \widehat{G} . In this case, the representation ring $R(G)$ can be naturally identified with $\mathbb{Z}[\widehat{G}]$. With this identification in mind, the $R(G)$ -module structure on $K_*(A \rtimes_\alpha G)$ can be described as follows (see Proposition 2.7.10 in [28]). Given τ in \widehat{G} and η in $K_*(A \rtimes_\alpha G)$, we have

$$\tau \cdot \eta = K_*(\widehat{\alpha}_\tau)(\eta).$$

Recall the following definition from [28].

Definition 2.4. Let G be a compact group. We let I_G denote the augmentation ideal of $R(G)$; that is, the kernel of the map $R(G) \rightarrow \mathbb{Z}$ which sends the class $[V]$ of a finite dimensional unitary representation V of G , to its dimension $\dim(V)$.

Given n in \mathbb{N} , we denote by I_G^n the product of I_G with itself n times.

2.2. Central sequence algebras. Let A be a unital C^* -algebra. Let $\ell^\infty(\mathbb{N}, A)$ denote the set of all bounded sequences $(a_n)_{n \in \mathbb{N}}$ in A , endowed with the supremum norm

$$\|(a_n)_{n \in \mathbb{N}}\| = \sup_{n \in \mathbb{N}} \|a_n\|$$

and pointwise operations. Then $\ell^\infty(\mathbb{N}, A)$ is a unital C^* -algebra, the unit being the constant sequence 1_A . Let

$$c_0(\mathbb{N}, A) = \left\{ (a_n)_{n \in \mathbb{N}} \in \ell^\infty(\mathbb{N}, A) : \lim_{n \rightarrow \infty} \|a_n\| = 0 \right\}.$$

Then $c_0(\mathbb{N}, A)$ is an ideal in $\ell^\infty(\mathbb{N}, A)$, and we denote the quotient

$$\ell^\infty(\mathbb{N}, A)/c_0(\mathbb{N}, A)$$

by A_∞ . Write $\kappa_A: \ell^\infty(\mathbb{N}, A) \rightarrow A_\infty$ for the quotient map. We identify A with the unital subalgebra of $\ell^\infty(\mathbb{N}, A)$ consisting of the constant sequences, and with a unital subalgebra of A_∞ by taking its image under κ_A . We write $A_\infty \cap A'$ for the relative commutant of A inside of A_∞ .

If $\alpha: G \rightarrow \text{Aut}(A)$ is an action of G on A , then there are actions of G on A_∞ and on $A_\infty \cap A'$, both denoted by α_∞ . Note that unless the group G is discrete, these actions will in general not be continuous. We set

$$\ell_\alpha^\infty(\mathbb{N}, A) = \{a \in \ell^\infty(\mathbb{N}, A) : g \mapsto (\alpha_\infty)_g(a) \text{ is continuous}\},$$

and $A_{\infty, \alpha} = \kappa_A(\ell_\alpha^\infty(\mathbb{N}, A))$. By construction, the restriction of α_∞ to $A_{\infty, \alpha}$ is continuous.

We note that if $\varphi: A \rightarrow B$ is a unital homomorphism of unital C^* -algebras, then φ induces unital homomorphisms $\ell^\infty(\mathbb{N}, \varphi): \ell^\infty(\mathbb{N}, A) \rightarrow \ell^\infty(\mathbb{N}, B)$ and $\varphi_\infty: A_\infty \rightarrow B_\infty$. The assignments $A \mapsto \ell^\infty(\mathbb{N}, A)$ and $A \mapsto A_\infty$ are functorial for unital C^* -algebras and unital homomorphisms.

Functoriality of the assignment $A \mapsto A_\infty \cap A'$ is more subtle, since not every map between C^* -algebras induces a map between the corresponding central sequences. In Lemma 2.5 and Lemma 2.6, we show two instances in which this is indeed the case. These two cases will be needed in the following section.

Lemma 2.5. Let $\varphi: A \rightarrow B$ be a surjective unital homomorphism of unital C^* -algebras. Then the restriction of φ_∞ to $A_\infty \cap A'$ induces a unital homomorphism $\varphi_\infty: A_\infty \cap A' \rightarrow B_\infty \cap B'$.

Moreover, if G is a locally compact group and $\alpha: G \rightarrow \text{Aut}(A)$ and $\beta: G \rightarrow$

$\text{Aut}(B)$ are continuous actions of G on A and B respectively, then φ also induces a unital homomorphism $\varphi_\infty: A_{\infty,\alpha} \cap A' \rightarrow B_{\infty,\beta} \cap B'$, which is equivariant if $\varphi: A \rightarrow B$ is.

Proof. We only need to check that $\varphi_\infty(A_\infty \cap A') \subseteq B_\infty \cap B'$. Let $a = (a_n)_{n \in \mathbb{N}}$ in $A_\infty \cap A'$ and let $b \in B$. We have to show that $\varphi_\infty(a)$ commutes with $\kappa_B(b)$. Choose $c \in A$ such that $\varphi(c) = b$. Since $\kappa_B \circ \varphi = \varphi_\infty \circ \kappa_A$, we have

$$[\varphi_\infty(a), b] = [\varphi_\infty(a), \kappa_B(\varphi(c))] = [a, \kappa_A(c)] = 0,$$

and the result follows.

The proof of the second claim is analogous. \square

An important case of when a unital homomorphism between unital C^* -algebras induces a unital homomorphism between the central sequence algebras is that of the unital inclusion $A \hookrightarrow A \otimes B$ of a unital C^* -algebra as the first tensor factor. This homomorphism is not covered by the previous lemma, so we shall prove it separately.

Lemma 2.6. Let A and B be unital C^* -algebras, let $A \otimes B$ be any C^* -algebra completion of the algebraic tensor product of A and B , and let $\iota: A \rightarrow A \otimes B$ be given by $\iota(a) = a \otimes 1$ for all $a \in A$. Then ι_∞ restricts to a unital homomorphism

$$\iota_\infty: A_\infty \cap A' \rightarrow (A \otimes B)_\infty \cap (A \otimes B)'$$

Moreover, if G is a locally compact group and $\alpha: G \rightarrow \text{Aut}(A)$ and $\beta: G \rightarrow \text{Aut}(B)$ are continuous actions of G on A and B respectively, and if the tensor product action

$$g \mapsto (\alpha \otimes \beta)_g = \alpha_g \otimes \beta_g$$

extends to $A \otimes B$, then ι induces a unital equivariant homomorphism

$$\iota_\infty: A_{\infty,\alpha} \cap A' \rightarrow (A \otimes B)_{\infty,\alpha \otimes \beta} \cap (A \otimes B)'$$

Proof. Let $a = (a_n)_{n \in \mathbb{N}}$ in $A_\infty \cap A'$ and let $x \in A \otimes B$. We may assume that x is a simple tensor, say $x = c \otimes b$ for some $c \in A$ and some $b \in B$. Then

$$[\iota_\infty(a), x] = [(a_n \otimes 1)_{n \in \mathbb{N}}, \kappa_{A \otimes B}(c \otimes b)] = 0,$$

since $\lim_{n \rightarrow \infty} \|[a_n, c]\| = 0$.

The proof of the second claim is straightforward. \square

2.3. Completely positive order zero maps. We briefly recall some of the basics of completely positive order zero maps. See [30] for more details and further results.

Let A be a C^* -algebra, and let a, b be elements in A . We say that a and b are *orthogonal*, and write $a \perp b$, if $ab = ba = a^*b = ab^* = 0$. If $a, b \in A$ are selfadjoint, then they are orthogonal if and only if $ab = 0$.

Definition 2.7. Let A and B be C^* -algebras, and let $\varphi: A \rightarrow B$ be a completely positive map. We say that φ has *order zero* if for every a and b in A , we have $\varphi(a) \perp \varphi(b)$ whenever $a \perp b$.

Remark 2.8. It is straightforward to check that C^* -algebra homomorphisms have order zero, and that the composition of two order zero maps is again order zero.

The following is the main result in [30].

Theorem 2.9. (Theorem 2.3 and Corollary 3.1 in [30]) Let A and B be C^* -algebras. There is a bijection between completely positive contractive order zero maps $A \rightarrow B$ and C^* -algebra homomorphisms $C_0((0,1]) \otimes A \rightarrow B$. A completely positive contractive order zero map $\varphi: A \rightarrow B$ induces the homomorphism $\rho_\varphi: C_0((0,1]) \otimes A \rightarrow B$ determined by $\rho_\varphi(\text{id}_{(0,1]} \otimes a) = \varphi(a)$ for all $a \in A$. Conversely, if $\rho: C_0((0,1]) \otimes A \rightarrow B$ is a homomorphism, then the induced completely positive contractive order zero map $\varphi_\rho: A \rightarrow B$ is the one given by $\varphi_\rho(a) = \rho(\text{id}_{(0,1]} \otimes a)$ for all $a \in A$.

The following easy corollary will be used throughout without reference.

Corollary 2.10. Let A and B be unital C^* -algebras, let G be a locally compact group, let $\alpha: G \rightarrow \text{Aut}(A)$ and $\beta: G \rightarrow \text{Aut}(B)$ be continuous actions of G on A and B respectively, and let $\varphi: A \rightarrow B$ be a completely positive order zero map. Denote by $\rho_\varphi: C_0((0,1]) \otimes A \rightarrow B$ the induced homomorphism given by Theorem 2.9. Give $C_0((0,1])$ the trivial action of G , and give $C_0((0,1]) \otimes A$ the corresponding diagonal action. Then φ is equivariant if and only if ρ_φ is equivariant.

Proof. We denote by $\tilde{\alpha}: G \rightarrow \text{Aut}(C_0((0,1]) \otimes A)$ the diagonal action described in the statement. Assume that ρ is equivariant. Given g in G and a in A , we have

$$\begin{aligned} \varphi_\rho(\tilde{\alpha}_g(\text{id}_{(0,1]} \otimes a)) &= \varphi_\rho(\text{id}_{(0,1]} \otimes \alpha_g(a)) \\ &= \rho(\alpha_g(a)) \\ &= \beta_g(\rho(a)) \\ &= \beta_g(\varphi_\rho(\text{id}_{(0,1]} \otimes a)). \end{aligned}$$

Since $\text{id}_{(0,1]}$ generates $C_0((0,1])$, we conclude that φ_ρ is equivariant.

Conversely, if φ_ρ is equivariant, it is clear that its restriction to the invariant tensor factor A is also equivariant. This finishes the proof. \square

3. ROKHLIN DIMENSION FOR COMPACT GROUP ACTIONS

We begin by recalling the definition of finite Rokhlin dimension for finite groups.

Definition 3.1. (See Definition 1.1 in [15].) Let G be a finite group, let A be a unital C^* -algebra, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of G on A . Given a non-negative integer d , we say that α has *Rokhlin dimension* d , and denote this by $\dim_{\text{Rok}}(\alpha) = d$, if d is the least integer with the following property: for every $\varepsilon > 0$ and for every finite subset F of A , there exist positive contractions $f_g^{(\ell)}$ for $g \in G$ and $\ell = 0, \dots, d$, satisfying the following conditions for every $\ell = 0, \dots, d$, for every $g, h \in G$, and for every $a \in F$:

- (1) $\left\| \alpha_h \left(f_g^{(\ell)} \right) - f_{hg}^{(\ell)} \right\| < \varepsilon;$
- (2) $\left\| f_g^{(\ell)} f_h^{(\ell)} \right\| < \varepsilon$ whenever $g \neq h;$
- (3) $\left\| \sum_{g \in G} \sum_{\ell=0, \dots, d} f_g^{(\ell)} - 1 \right\| < \varepsilon;$
- (4) $\left\| \left[f_g^{(\ell)}, a \right] \right\| < \varepsilon.$

If one can always choose the positive contractions $f_g^{(\ell)}$ above to moreover satisfy

$$\left\| \left[f_g^{(\ell)}, f_h^{(k)} \right] \right\| < \varepsilon$$

for every $g, h \in G$ and every $k, \ell = 0, \dots, d$, then we say that α has *Rokhlin dimension d with commuting towers*, and denote this by $\dim_{\text{Rok}}^c(\alpha) = d$.

Given a compact group G , we denote by $\text{Lt}: G \rightarrow \text{Homeo}(G)$ the action of left translation. With a slight abuse of notation, we will also denote by Lt the induced action of G on $C(G)$.

Definition 3.1 can be generalized to the case of second countable compact groups as follows.

Definition 3.2. Let G be a second countable, Hausdorff compact group, let A be a unital C^* -algebra, and let $\alpha: G \rightarrow \text{Aut}(A)$ be a continuous action. We say that α has *Rokhlin dimension d* , if d is the least integer such that there exist equivariant completely positive contractive order zero maps

$$\varphi_0, \dots, \varphi_d: (C(G), \text{Lt}) \rightarrow (A_{\infty, \alpha} \cap A', \alpha_\infty)$$

such that $\varphi_0(1) + \dots + \varphi_d(1) = 1$.

We denote the Rokhlin dimension of α by $\dim_{\text{Rok}}(\alpha)$. If no integer d as above exists, we say that α has *infinite Rokhlin dimension*, and denote it by $\dim_{\text{Rok}}(\alpha) = \infty$. If one can always choose the maps $\varphi_0, \dots, \varphi_d$ to have commuting ranges, then we say that α has *Rokhlin dimension d with commuting towers*, and write $\dim_{\text{Rok}}^c(\alpha) = d$.

Remark 3.3. It is an easy exercise to check that if G is a finite group, then Definition 3.2 agrees with Definition 1.1 in [15].

It is clear that if A is commutative, then the notions of Rokhlin dimension with and without commuting towers agree. Nevertheless, Example 4.8 below shows that commuting towers cannot always be arranged, even for \mathbb{Z}_2 -actions on \mathcal{O}_2 with Rokhlin dimension 1. In fact, it seems that there really is a big difference between these two notions, although we do not know how much they differ in general.

Remark 3.4. It follows from Theorem 2.3 in [30] that a unital completely positive contractive order zero map is necessarily a homomorphism. In particular, Rokhlin dimension zero is equivalent to the Rokhlin property as in Definition 2.3 of [14]. (We point out that the requirement that φ be injective in Definition 2.3 of [14] is unnecessary: its kernel is a translation-invariant ideal of $C(G)$, so it must be either $\{0\}$ or $C(G)$. Since φ is assumed to be unital, it must be $\ker(\varphi) = \{0\}$.)

The following result is probably well-known to the experts. Since we have not been able to find a reference, we present its proof here.

Lemma 3.5. Let A and B be C^* -algebras, and let $\varphi: A \rightarrow B$ be a completely positive contractive order zero map. Then

$$\ker(\varphi) = \{a \in A: \varphi(a) = 0\}$$

is a closed two-sided ideal in A .

Proof. That $\ker(\varphi)$ is closed follows easily by continuity of φ . Let us now show that it is a two-sided ideal.

Let $\pi: C_0((0, 1]) \otimes A \rightarrow B$ be the homomorphism determined by $\pi(\text{id}_{(0, 1]} \otimes a) = \varphi(a)$ for all $a \in A$ (see Theorem 2.9 above). Then $\ker(\pi)$ is an ideal of $C_0((0, 1]) \otimes A$. Let $a \in \ker(\varphi)$ and let $x \in A$, and assume that a is positive. Then ax belongs to $\ker(\varphi)$ if and only if $\text{id}_{(0, 1]} \otimes ax$ belongs to $\ker(\pi)$. Denote by $t^{1/2}$ the map

$(0, 1] \rightarrow (0, 1]$ given by $x \mapsto \sqrt{x}$. By functional calculus, $t^{1/2} \otimes a^{1/2}$ belongs to $\ker(\pi)$. It follows that

$$\text{id}_{(0,1]} \otimes ax = \left(t^{1/2} \otimes a^{1/2}\right) \left(t^{1/2} \otimes a^{1/2}x\right) \in \ker(\pi)$$

since $\ker(\pi)$ is an ideal in $C_0((0, 1]) \otimes A$, and hence $\varphi(ax) = \pi(\text{id}_{(0,1]} \otimes ax) = 0$. A similar argument shows that $\varphi(xa) = 0$ as well, proving that $\ker(\varphi)$ is a two-sided ideal in A . \square

Corollary 3.6. Adopt the notation of Definition 3.2 above. Then the order zero maps $\varphi_0, \dots, \varphi_d$ are either zero or injective.

Proof. For $j = 0, \dots, d$, the kernel I_j of φ_j is a translation invariant ideal in $C(G)$, since φ_j is equivariant. The result now follows from the fact that the only translation invariant ideals of $C(G)$ are $\{0\}$ and $C(G)$. \square

In particular, if $\dim_{\text{Rok}}(\alpha) = d < \infty$, then the maps $\varphi_0, \dots, \varphi_d$ from Definition 3.2 are injective.

We start by presenting some permanence properties for actions of compact groups with finite Rokhlin dimension. Not surprisingly, finite Rokhlin dimension is far more flexible than the Rokhlin property, and it is preserved by several constructions. Most notably, finite Rokhlin dimension for finite dimensional compact groups (in particular, for Lie groups) is inherited by the restriction to any closed subgroup, except that the actual dimension may increase.

We begin with a technical lemma which characterizes finite Rokhlin dimension in terms of elements in the C^* -algebra itself, rather than its central sequence algebra.

Lemma 3.7. Let G be a compact group, let A be a separable unital C^* -algebra, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of G on A . Let d be a non-zero integer.

- (1) We have $\dim_{\text{Rok}}(\alpha) \leq d$ if and only if for every $\varepsilon > 0$, for every finite subset S of $C(G)$, and for every compact subset F of A , there exist completely positive contractive maps $\psi_0, \dots, \psi_d: C(G) \rightarrow A$ satisfying the following conditions:
 - (a) $\|\psi_j(f)a - a\psi_j(f)\| < \varepsilon$ for all $j = 0, \dots, d$, for all f in S , and all a in F .
 - (b) $\|\psi_j(\text{Lt}_g(f)) - \alpha_g(\psi_j(f))\| < \varepsilon$ for all $j = 0, \dots, d$, for all g in G , and for all f in S .
 - (c) $\|\psi_j(f_1)\psi_j(f_2)\| < \varepsilon$ whenever f_1 and f_2 in S are orthogonal.
 - (d) $\left\| \sum_{j=0}^d \psi_j(1_{C(G)}) - 1_A \right\| < \varepsilon$.
- (2) We have $\dim_{\text{Rok}}^c(\alpha) \leq d$ if and only if for every $\varepsilon > 0$, for every finite subset S of $C(G)$, and for every compact subset F of A , there exist completely positive contractive maps $\psi_0, \dots, \psi_d: C(G) \rightarrow A$ satisfying the conditions listed above in addition to

$$\|\psi_j(f_1)\psi_k(f_2) - \psi_k(f_2)\psi_j(f_1)\| < \varepsilon$$

for all $j, k = 0, \dots, d$ and all f_1 and f_2 in S .

Proof. We prove (1) first. Assume that for every $\varepsilon > 0$, for every finite subset S of $C(G)$, and every finite subset F of A , there exist completely positive contractive

maps $\varphi_0, \dots, \varphi_d: C(G) \rightarrow A$ satisfying the conditions of the statement. Choose increasing sequences $(F_n)_{n \in \mathbb{N}}$ and $(S_n)_{n \in \mathbb{N}}$ of finite subsets of A and $C(G)$, whose union is dense in A and in $C(G)$, respectively. Let $\psi_0^{(n)}, \dots, \psi_d^{(n)}: C(G) \rightarrow A$ be as in the statement for the choices F_n and $\frac{1}{n}$. For $j = 0, \dots, d$, denote by $\varphi_j: C(G) \rightarrow A_\infty$ the linear map given by

$$\varphi_j(\kappa_A((a_n)_{n \in \mathbb{N}})) = \kappa_A \left(\left(\psi_j^{(n)}(a_n) \right)_{n \in \mathbb{N}} \right)$$

for all $(a_n)_{n \in \mathbb{N}}$ in $\ell^\infty(\mathbb{N}, A)$. Then φ_j is easily seen to be completely positive contractive and order zero. It is also straightforward to check that its image is contained in $A_{\infty, \alpha} \cap A'$, and that it is equivariant. Finally, it is immediate that $\sum_{j=0}^d \varphi_j(1) = 1$.

Conversely, suppose that α has Rokhlin dimension at most d . Choose completely positive contractive order zero maps $\varphi_0, \dots, \varphi_d: C(G) \rightarrow A_{\infty, \alpha} \cap A'$ as in the definition of finite Rokhlin dimension. Fix j in $\{0, \dots, d\}$. By Choi-Effros, there exist completely positive contractive maps $\psi_j = (\psi_j^{(n)})_{n \in \mathbb{N}}: C(G) \rightarrow \ell^\infty(\mathbb{N}, A)$ for $j = 0, \dots, d$ such that for all f, f_1, f_2 in $C(G)$ with f_1 orthogonal to f_2 , for all a in A , and for all g in G , we have

$$\begin{aligned} \left\| \psi_j^{(n)}(f)a - a\psi_j^{(n)}(f) \right\| &\rightarrow 0 \\ \left\| \psi_j^{(n)}(\text{Lt}_g(f)) - \alpha_g(\psi_j^{(n)}(f)) \right\| &\rightarrow 0 \\ \left\| \psi_j^{(n)}(f_1)\psi_j^{(n)}(f_2) \right\| &\rightarrow 0 \\ \left\| \sum_{j=0}^d \psi_j^{(n)}(1_{C(G)}) - 1_A \right\| &\rightarrow 0 \end{aligned}$$

Given $\varepsilon > 0$, given a finite subset S of $C(G)$ and given a finite subset F of A , choose a positive integer n such that the quantities above are all less than ε on the elements of S and F , respectively, and set $\psi_j = \psi_j^{(n)}$ for $j = 0, \dots, d$. This finishes the proof of (1).

The proof of (2) is analogous. In particular, in the “only if” implication, one has to use that the completely positive contractive maps $\psi_j = (\psi_j^{(n)})_{n \in \mathbb{N}}: C(G) \rightarrow \ell^\infty(\mathbb{N}, A)$ for $j = 0, \dots, d$ obtained from Choi-Effros, moreover satisfy

$$\lim_{n \rightarrow \infty} \left\| \psi_j^{(n)}(f_1)\psi_k^{(n)}(f_2) - \psi_k^{(n)}(f_2)\psi_j^{(n)}(f_1) \right\| = 0$$

for all f_1 and f_2 in $C(G)$, and all $j, k = 0, \dots, d$. We omit the details. \square

Regarding finite Rokhlin dimension as a noncommutative analog of freeness of group actions on topological spaces, we give the following interpretation of Theorem 3.8 below. Part (1) is the analog of the fact that a diagonal action on a product space is free if only of the factors is free; part (2) is the analog of the fact that the restriction of a free action to an invariant closed subset is also free; and part (3) is the analog of the fact that an inverse limit of free actions is again free.

Theorem 3.8. Let A be a unital C^* -algebra let G be a compact group, and let $\alpha: G \rightarrow \text{Aut}(A)$ be a continuous action of G on A .

- (1) Let B be a unital C^* -algebra, and let $\beta: G \rightarrow \text{Aut}(B)$ be a continuous action of G on B . Let $A \otimes B$ be any C^* -algebra completion of the algebraic tensor product of A and B for which the tensor product action $g \mapsto (\alpha \otimes \beta)_g = \alpha_g \otimes \beta_g$ is defined. Then

$$\dim_{\text{Rok}}(\alpha \otimes \beta) \leq \min \{ \dim_{\text{Rok}}(\alpha), \dim_{\text{Rok}}(\beta) \}$$

and

$$\dim_{\text{Rok}}^c(\alpha \otimes \beta) \leq \min \{ \dim_{\text{Rok}}^c(\alpha), \dim_{\text{Rok}}^c(\beta) \}.$$

- (2) Let I be an α -invariant ideal in A , and denote by $\bar{\alpha}: G \rightarrow \text{Aut}(A/I)$ the induced action on the quotient. Then

$$\dim_{\text{Rok}}(\bar{\alpha}) \leq \dim_{\text{Rok}}(\alpha)$$

and

$$\dim_{\text{Rok}}^c(\bar{\alpha}) \leq \dim_{\text{Rok}}^c(\alpha).$$

Furthermore,

- (3) Let $(A_n, \iota_n)_{n \in \mathbb{N}}$ be a direct system of unital C^* -algebras with unital connecting maps, and for each $n \in \mathbb{N}$, let $\alpha^{(n)}: G \rightarrow \text{Aut}(A_n)$ be a continuous action such that $\iota_n \circ \alpha_g^{(n)} = \alpha_g^{(n+1)} \circ \iota_n$ for all $n \in \mathbb{N}$ and all $g \in G$. Suppose that $A = \varinjlim A_n$ and $\alpha = \varinjlim \alpha^{(n)}$. Then

$$\dim_{\text{Rok}}(\alpha) \leq \liminf_{n \rightarrow \infty} \dim_{\text{Rok}}(\alpha^{(n)})$$

and

$$\dim_{\text{Rok}}^c(\alpha) \leq \liminf_{n \rightarrow \infty} \dim_{\text{Rok}}^c(\alpha^{(n)}).$$

Proof. We only prove the results for the noncommuting tower version; the proofs for the commuting tower version are analogous and are left to the reader.

Part (1). The statement is immediate if both α and β have infinite Rokhlin dimension, so assume that $\dim_{\text{Rok}}(\alpha) = d < \infty$. Then there are equivariant completely positive contractive order zero maps

$$\varphi_0, \dots, \varphi_d: C(G) \rightarrow A_{\infty, \alpha} \cap A',$$

such that $\varphi_0(1) + \dots + \varphi_d(1) = 1$. Denote by $\iota: A \rightarrow A \otimes B$ the canonical embedding as the first tensor factor. By Lemma 2.6, this inclusion induces a unital homomorphism $\iota_{\infty}: A_{\infty} \cap A' \rightarrow (A \otimes B)_{\infty, \alpha \otimes \beta} \cap (A \otimes B)'$, and ι_{∞} is moreover equivariant with respect to α_{∞} and $(\alpha \otimes \beta)_{\infty}$. For $j = 0, \dots, d$, set

$$\psi_j = \iota_{\infty} \circ \varphi_j: C(G) \rightarrow (A \otimes B)_{\infty, \alpha \otimes \beta} \cap (A \otimes B)'.$$

Then ψ_0, \dots, ψ_d are equivariant completely positive contractive order zero maps, and $\psi_0(1) + \dots + \psi_d(1) = \varphi_0(1) + \dots + \varphi_d(1) = 1$. Hence $\dim_{\text{Rok}}(\alpha \otimes \beta) \leq d$ and the result follows.

Part (2). The statement is immediate if α has infinite Rokhlin dimension, so suppose that there exist a positive integer d in \mathbb{N} and equivariant completely positive contractive order zero maps

$$\varphi_0, \dots, \varphi_d: C(G) \rightarrow A_{\infty, \alpha} \cap A'$$

such that $\varphi_0(1) + \dots + \varphi_d(1) = 1$. Denote by $\pi: A \rightarrow A/I$ the quotient map. Lemma 2.5 implies that π induces a unital homomorphism

$$\pi_{\infty}: A_{\infty} \cap A' \rightarrow (A/I)_{\infty, \bar{\alpha}} \cap (A/I)'.$$

Moreover, this homomorphism is easily seen to be equivariant. For $j = 0, \dots, d$, set $\psi_j = \pi_\infty \circ \varphi_j: C(G) \rightarrow (A/I)_{\infty, \overline{\alpha}} \cap (A/I)'$. Then ψ_j is an equivariant completely positive contractive order zero map for all $j = 0, \dots, d$, and $\psi_0(1) + \dots + \psi_d(1) = \varphi_0(1) + \dots + \varphi_d(1) = 1$. It follows that $\dim_{\text{Rok}}(\overline{\alpha}) \leq d$, as desired.

Part (3). The statement is immediate if $\liminf_{n \rightarrow \infty} \dim_{\text{Rok}}(\alpha^{(n)}) = \infty$. We shall therefore assume that there exists $d \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, there is $n \geq m$ in \mathbb{N} with $\dim_{\text{Rok}}(\alpha^{(n)}) \leq d$. By passing to a subsequence, we may also assume that $\dim_{\text{Rok}}(\alpha^{(n)}) \leq d$ for all $n \in \mathbb{N}$.

We use Lemma 3.7. Let $\varepsilon > 0$, let S be a finite subset of $C(G)$, and let F be a finite subset of A . With $L = \text{card}(F)$, write $F = \{a_1, \dots, a_L\}$, and find a positive integer n in \mathbb{N} and elements b_1, \dots, b_L in A_n such that $\|a_j - b_j\| < \frac{\varepsilon}{2}$ for all $j = 1, \dots, L$. Choose completely positive contractive maps $\psi_0, \dots, \psi_d: C(G) \rightarrow A_n$ satisfying conditions (a) through (d) in part (1) of Lemma 3.7 for $\frac{\varepsilon}{2}$ and the finite set $F' = \{b_1, \dots, b_L\} \subseteq A_n$. If $\iota_{n, \infty}: A_n \rightarrow A$ denotes the canonical map, then it is easy to check that the completely positive contractive maps

$$\iota_{n, \infty} \circ \psi_0, \dots, \iota_{n, \infty} \circ \psi_d: C(G) \rightarrow A$$

satisfy conditions (a) through (d) in part (1) of Lemma 3.7 for ε and the finite set F . This shows the result in the case of non commuting towers.

The proof in the commuting towers case is analogous, using also the extra condition in part (2) of Lemma 3.7. We omit the details. \square

We point out that finite Rokhlin dimension does not pass to invariant subalgebras, even if the original action has the Rokhlin property, the C^* -algebra is \mathcal{O}_2 , the invariant subalgebra is isomorphic to \mathcal{O}_2 and the action restricted to the invariant subalgebra is pointwise outer, as the next example shows.

Example 3.9. Denote by s_1 and s_2 the canonical generators of the Cuntz algebra \mathcal{O}_2 . Let $\gamma: \mathbb{T} \rightarrow \text{Aut}(\mathcal{O}_2)$ be the gauge action, this is, the action determined by $\gamma_\zeta(s_j) = \zeta s_j$ for all $\zeta \in \mathbb{T}$ and for $j = 1, 2$. Choose an isomorphism $\varphi: \mathcal{O}_2 \otimes \mathcal{O}_2 \rightarrow \mathcal{O}_2$ and let $\alpha: \mathbb{T} \rightarrow \text{Aut}(\mathcal{O}_2)$ be given by

$$\alpha_\zeta = \varphi \circ (\gamma_\zeta \otimes \text{id}_{\mathcal{O}_2}) \circ \varphi^{-1}$$

for all $\zeta \in \mathbb{T}$. It is shown in Example 3.7 in [5] that α has the Rokhlin property. On the other hand, it is immediate that the restriction of α to the invariant subalgebra $\varphi(1 \otimes \mathcal{O}_2)$ does not have finite Rokhlin dimension, since it is conjugate to the trivial action on \mathcal{O}_2 . We claim that the restriction of α to the invariant subalgebra $\varphi(\mathcal{O}_2 \otimes 1)$, does not have finite Rokhlin dimension with commuting towers. Said restriction is conjugate to the gauge action γ on \mathcal{O}_2 . To show that $\dim_{\text{Rok}}^c(\gamma) = \infty$, it is enough to show that no power of the augmentation ideal $I_{\mathbb{T}}$ annihilates $K_*^{\mathbb{T}}(\mathcal{O}_2)$. Recall that the crossed product of \mathcal{O}_2 by the gauge action is isomorphic to $M_{2^\infty} \otimes \mathcal{K}$. For n in \mathbb{N} , and under the canonical identifications given by Julg's Theorem (here reproduced as Theorem 2.3), we have

$$I_{\mathbb{T}}^n \cdot K_*^{\mathbb{T}}(\mathcal{O}_2) \cong \text{Im} \left((\text{id}_{K_0(M_{2^\infty} \otimes \mathcal{K})} - K_*(\hat{\gamma}))^n \right).$$

It is a well known fact that $\hat{\gamma}$ is the unilateral shift on $M_{2^\infty} \otimes \mathcal{K}$, whose induced action on K_0 is multiplication by 2. Thus $\text{id}_{K_0(M_{2^\infty} \otimes \mathcal{K})} - K_0(\hat{\gamma})$ is multiplication by -1, which is an isomorphism of $\mathbb{Z}[\frac{1}{2}] \cong K_*^{\mathbb{T}}(\mathcal{O}_2)$. In particular, any of its powers

is also an isomorphism, and thus

$$I_{\mathbb{T}}^n \cdot K_0^{\mathbb{T}}(\mathcal{O}_2) = K_0^{\mathbb{T}}(\mathcal{O}_2) \neq \{0\}$$

for all n in \mathbb{N} . This proves the claim.

3.1. Restrictions to closed subgroups. We now turn to restrictions of actions in relation to Rokhlin dimension. The following result is the analog of the fact that the restriction of a free action to a (closed) subgroup is again free.

Theorem 3.10. Let A be a unital C^* -algebra, let G be a finite dimensional compact group, let H be a closed subgroup of G , and let $\alpha: G \rightarrow \text{Aut}(A)$ be a continuous action. Then

$$\dim_{\text{Rok}}(\alpha|_H) \leq (\dim(G) - \dim(H) + 1)(\dim_{\text{Rok}}(\alpha) + 1) - 1$$

and

$$\dim_{\text{Rok}}^c(\alpha|_H) \leq (\dim(G) - \dim(H) + 1)(\dim_{\text{Rok}}^c(\alpha) + 1) - 1$$

Proof. Without loss of generality, we may assume that $\dim_{\text{Rok}}(\alpha)$ is finite.

Being a closed subspace of a finite dimensional subspace, H is finite dimensional. Let $d = \dim(G/H) = \dim(G) - \dim(H)$. We will produce $d+1$ completely positive contractive H -equivariant order zero maps $\varphi_0, \dots, \varphi_d: C(H) \rightarrow C(G)$ with $\varphi_0(1) + \dots + \varphi_d(1) = 1$. Once we have done this, and since these maps will obviously have commuting ranges, both claims will follow by composing each of the maps $\varphi_0, \dots, \varphi_d$ with the $\dim_{\text{Rok}}(\alpha) + 1$ maps as in the definition of finite Rokhlin dimension for α . The result will then be $(d+1)(\dim_{\text{Rok}}(\alpha) + 1)$ maps which will satisfy the definition of finite Rokhlin dimension for $\alpha|_H$.

Denote by $\pi: G \rightarrow G/H$ the canonical surjection. By part (1) of Theorem 2 in [19], the map $\pi: G \rightarrow G/H$ is a principal H -bundle. In particular, there exist local cross-sections from the orbit space G/H to G . For every $x \in G$, let $V_{\pi(x)}$ be a neighborhood of $\pi(x)$ in G/H where π is trivial. Using compactness of G/H , let \mathcal{U} be a finite subcover of G/H . Use Proposition 1.5 in [22] to refine \mathcal{U} to a d -decomposable covering \mathcal{V} . In other words, \mathcal{V} can be written as the disjoint union of $d+1$ families $\mathcal{V}_0 \cup \dots \cup \mathcal{V}_d$ of open sets, in such a way that for every $k = 0, \dots, d$, the elements of \mathcal{V}_k are pairwise disjoint.

For $k = 0, \dots, d$, let V_k denote the union of all the open sets in \mathcal{V}_k , and note that there is a cross-section defined on V_k . Let $\{f_0, \dots, f_d\}$ be a partition of unity of G/H subordinate to the cover $\{V_0, \dots, V_d\}$. Upon replacing V_k with the open set $f_k^{-1}((0, 1]) \subseteq V_k$, we may assume that $V_k = f_k^{-1}((0, 1])$ for all $k = 0, \dots, d$. For each $k = 0, \dots, d$, set $U_k = \pi^{-1}(V_k) \subseteq G$, and observe that there is an equivariant homeomorphism $U_k \cong V_k \times H$, where the H -action on $V_k \times H$ is diagonal with the trivial action on V_k and translation on H . Define a continuous function $\phi_k: V_k \times H \cong U_k \rightarrow (0, 1] \times H$ by

$$\phi_k(x, h) = (f_k(x), h)$$

for all (x, h) in $V_k \times H \cong U_k$. Then ϕ_k is continuous because the cross-section is continuous. Moreover, ϕ_k is clearly equivariant.

Identify $C_0((0, 1]) \otimes C(H)$ with $C_0((0, 1] \times H)$, and for $k = 0, \dots, d$ define

$$\psi_k: C_0((0, 1]) \otimes C(H) \rightarrow C(G)$$

by

$$\psi_k(f)(x) = \begin{cases} (f \circ \phi_k)(x), & \text{if } x \in U_k; \\ 0, & \text{else.} \end{cases}$$

Then ψ_k is a homomorphism, and it is equivariant since ϕ_k is. The map $\varphi_k: C(H) \rightarrow C(G)$ given by $\varphi_k(f) = \psi_k(\text{id}_{(0,1]} \otimes f)$ for $f \in C(H)$ is an equivariant completely positive contractive order zero map. Finally, using that $(f_k)_{k=0}^d$ is a partition of unity of G/H at the last step, we have

$$\sum_{k=0}^d \varphi_k(1) = \sum_{k=0}^d \psi_k(\text{id}_{(0,1]} \otimes 1) = \sum_{k=0}^d f_k = 1.$$

It follows that the maps $\varphi_0, \dots, \varphi_d$ have the desired properties, and the proof is finished. \square

In some cases, restricting to a subgroup does not increase the Rokhlin dimension. In the following proposition, the group is not assumed to be finite dimensional.

Proposition 3.11. Let A be a unital C^* -algebra, let G be a compact group, and let $\alpha: G \rightarrow \text{Aut}(A)$ be a continuous action. Let H be a closed subgroup of G , and assume that at least one of the following holds:

- (1) the coset space G/H is zero dimensional (this is the case whenever H has finite index in G).
- (2) $G = \prod_{i \in I} G_i$ or $G = \bigoplus_{i \in I} G_i$, and $H = G_j$ for some $j \in I$.
- (3) H is the connected component of G containing its unit.

Then

$$\dim_{\text{Rok}}(\alpha|_H) \leq \dim_{\text{Rok}}(\alpha) \quad \text{and} \quad \dim_{\text{Rok}}^c(\alpha|_H) \leq \dim_{\text{Rok}}^c(\alpha).$$

Proof. In all these cases, we will produce a unital H -equivariant homomorphism $C(H) \rightarrow C(G)$, where the H action on both $C(H)$ and $C(G)$ is given by left translation. This is easily seen to be equivalent to the existence of a continuous map $\phi: G \rightarrow H$ such that $\phi(hg) = h\phi(g)$ for all $h \in H$ and all $g \in G$.

Assuming the existence of such a homomorphism $C(H) \rightarrow C(G)$, the result will follow by composing it with the completely positive contractive order zero maps associated with α , similarly to what was done in parts (1) and (2) of Theorem 3.8.

(1). Assume that G/H is zero-dimensional. By Theorem 8 in [25], there exists a continuous section $\lambda: G/H \rightarrow G$. Denote by $\pi: G \rightarrow G/H$ the quotient map, and define $\phi: G \rightarrow H$ by

$$\phi(g) = g(\lambda(\pi(g)))^{-1}$$

for all $g \in G$. We check that the range of ϕ , which a priori is contained in G , really lands in H :

$$\pi(\phi(g)) = \pi(g)\pi(\lambda(\pi(g))^{-1}) = \pi(g)\pi(g)^{-1} = 1$$

for all $g \in G$, so $\phi(G) \subseteq H$. Continuity of ϕ follows from continuity of λ and from continuity of the group operations on G . Finally, if $h \in H$ and $g \in G$, then

$$\phi(hg) = hg\lambda(\pi(hg))^{-1} = hg\lambda(\pi(g))^{-1} = h\phi(g),$$

as desired.

(2). Both cases follow from the fact that there is a group homomorphism $G \rightarrow G_j$ determined by $(g_i)_{i \in I} \mapsto g_j$.

(3). This follows from (1) and the fact that G/G_0 is totally disconnected. \square

Rokhlin dimension can indeed increase when passing to a subgroup, even if the original action has the Rokhlin property.

Example 3.12. Let $\alpha: \mathbb{T} \rightarrow \text{Aut}(C(\mathbb{T}))$ be given by $\alpha_\zeta(f)(\omega) = f(\zeta^{-1}\omega)$ for $\zeta, \omega \in \mathbb{T}$ and $f \in C(\mathbb{T})$. Then α has Rokhlin dimension zero. Given $n \in \mathbb{N}$ with $n > 1$, identify \mathbb{Z}_n with the subgroup of \mathbb{T} consisting of the n -th roots of unity. Then

$$\dim_{\text{Rok}}^c(\alpha|_{\mathbb{Z}_n}) = \dim_{\text{Rok}}(\alpha|_{\mathbb{Z}_n}) = 1.$$

Indeed, $\dim_{\text{Rok}}(\alpha|_{\mathbb{Z}_n}) \leq 1$ by Theorem 3.10. If $\dim_{\text{Rok}}(\alpha|_{\mathbb{Z}_n}) = 0$, then $\alpha|_{\mathbb{Z}_n}$ would have the Rokhlin property, which in particular would imply the existence of a non-trivial projection in $C(S^1)$, which is a contradiction.

Even for circle actions with the Rokhlin property, there are less obvious K -theoretic obstructions for the restriction of a circle action with the Rokhlin property to have the Rokhlin property, besides merely the lack of projections, as the next example shows.

Example 3.13. (See Example 4.19 in [5]) There is an example of a purely infinite simple separable nuclear unital C^* -algebra, and an action of the circle on it with the Rokhlin property, such that no restriction to a finite subgroup of \mathbb{T} has the Rokhlin property.

To construct it, let $\{p_n\}_{n \in \mathbb{N}}$ be an enumeration of the prime numbers, and for every $n \in \mathbb{N}$ set $q_n = p_1 \cdots p_n$. Fix a countable dense subset $X = \{x_1, x_2, x_3, \dots\}$ of \mathbb{T} with $x_1 = 1$. For $n \in \mathbb{N}$, define a unital injective map $\iota_n: M_{q_n}(C(\mathbb{T})) \rightarrow M_{q_{n+1}}(C(\mathbb{T}))$ by

$$\iota_n(f) = \begin{pmatrix} f & 0 & \cdots & 0 \\ 0 & \text{Lt}_{x_2}(f) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \text{Lt}_{x_{p_n}}(f) \end{pmatrix}$$

for f in $M_{q_n}(C(\mathbb{T}))$. The direct limit $A = \varinjlim (M_{q_n}(C(\mathbb{T})), \iota_n)$ is a simple unital AT-algebra. For $n \in \mathbb{N}$, let $\alpha^{(n)}: \mathbb{T} \rightarrow \text{Aut}(M_{q_n}(C(\mathbb{T})))$ be the tensor product of the trivial action on M_{q_n} with the action of left translation on $C(\mathbb{T})$. Then the sequence $(\alpha^{(n)})_{n \in \mathbb{N}}$ induces a direct limit action $\alpha = \varinjlim \alpha^{(n)}$ of \mathbb{T} on A .

Now set $B = A \otimes \mathcal{O}_\infty$ and define $\beta: \mathbb{T} \rightarrow \text{Aut}(B)$ by $\beta = \alpha \otimes \text{id}_{\mathcal{O}_\infty}$. Then B is a purely infinite, simple, separable, nuclear unital C^* -algebra satisfying the UCT. It is shown in Example 4.19 in [5] that β has the Rokhlin property, and that for every $m \in \mathbb{N}$ with $m > 1$, the restriction $\beta|_m: \mathbb{Z}_m \rightarrow \text{Aut}(B)$ does not have the Rokhlin property.

We recall a result from [5] that gives a sufficient (and many times also necessary) condition for the restriction of an action with the Rokhlin property to have the Rokhlin property.

Theorem 3.14. (Theorem 7.18 of [5]) Let A be a separable unital C^* -algebra, let $n \in \mathbb{N}$ and let $\alpha: \mathbb{T} \rightarrow \text{Aut}(A)$ be an action with the Rokhlin property. Suppose that A absorbs M_{n^∞} . Then $\alpha|_{\mathbb{Z}_n}: \mathbb{Z}_n \rightarrow \text{Aut}(A)$ has the Rokhlin property.

We finish the discussion about restrictions of actions with finite Rokhlin dimension with an example that shows that there is in general no way to determine the Rokhlin dimension of an action only by knowing the Rokhlin dimension of all of its restrictions, at least in the case of commuting towers. Our example is somewhat

surprising: we exhibit an example of a circle action on a C^* -algebra, such that all of its restrictions to finite subgroups have the Rokhlin property, but the action itself has infinite Rokhlin dimension with commuting towers. (We do not know whether this action has finite Rokhlin dimension with non-commuting towers.)

Example 3.15. (See Example 5.12 in [5].) Let A be the universal UHF-algebra, this is, $A = \varinjlim (M_{n!}, \iota_n)$ where $\iota_n: M_{n!} \rightarrow M_{(n+1)!}$ is given by $\iota_n(a) = \text{diag}(a, \dots, a)$ for all a in $M_{n!}$. For every $n \in \mathbb{N}$, let $\alpha^{(n)}: \mathbb{T} \rightarrow \text{Aut}(M_{n!})$ be given by

$$\alpha_\zeta^{(n)} = \text{Ad}(\text{diag}(1, \zeta, \dots, \zeta^{n!-1}))$$

for all $\zeta \in \mathbb{T}$. Then $\iota_n \circ \alpha_\zeta^{(n)} = \alpha_\zeta^{(n+1)} \circ \iota_n$ for all $n \in \mathbb{N}$ and all $\zeta \in \mathbb{T}$, and hence there is a direct limit action $\alpha = \varinjlim \alpha^{(n)}$ of \mathbb{T} on A . It is shown in Example 5.12 in [5] that for every positive integer $m > 1$, the restriction $\alpha|_{\mathbb{Z}_m}: \mathbb{Z}_m \rightarrow \text{Aut}(A)$ has the Rokhlin property, and that the action α itself does not have the Rokhlin property. It moreover follows from Corollary 4.18 that α does not even have finite Rokhlin dimension with commuting towers.

Example 3.15 should be contrasted with the following fact.

Proposition 3.16. Let a compact Lie group G act on a locally compact Hausdorff space X . Then the action is free if and only if its restriction to every finite cyclic subgroup of G of *prime order* is free.

Proof. The “only if” implication is immediate. For the “if” implication, let $g \in G \setminus \{1\}$ and assume that there exists x in X with $gx = x$. The stabilizer subgroup

$$S_x = \{h \in G: hx = x\}$$

of x is therefore non-trivial. Being a closed subgroup of G , it is a Lie group by Cartan’s theorem. It follows that S_x has a finite cyclic group of prime order: this is immediate if S_x is finite, while if S_x is infinite, it must contain a (maximal) torus. Now, the restriction of the action to any such subgroup is trivial, contradicting the assumption. It follows that the action of G on X is free. \square

4. COMPARISON WITH OTHER NOTIONS OF NON-COMMUTATIVE FREENESS

In this section, we compare the notion of finite Rokhlin dimension (with and without commuting towers) with some of the other forms of freeness of group actions on C^* -algebras that have been studied. The properties we discuss here include freeness of actions on compact Hausdorff spaces in the commutative case, the Rokhlin property, discrete K -theory, local discrete K -theory, total K -freeness, and pointwise outerness.

We begin by comparing finite Rokhlin dimension on unital commutative C^* -algebras with freeness of the induced action on the maximal ideal space. Notice that in the case of commutative C^* -algebras, the distinction between commuting and non-commuting towers is irrelevant.

Lemma 4.1. Let G be a compact group acting on a compact Hausdorff space X . Denote by $\alpha: G \rightarrow \text{Aut}(C(X))$ the induced action of G on X and let n be a non-negative integer. Then α has Rokhlin dimension at most n if and only if there are an open cover $\{U_0, \dots, U_n\}$ of $(\beta\mathbb{N} \setminus \mathbb{N}) \times X$ consisting of G -invariant open sets, and continuous, proper, equivariant functions $\phi_j: U_j \rightarrow G \times (0, 1]$, where the action of G on $G \times (0, 1]$ is translation on G and trivial on $(0, 1]$.

Proof. Note that

$$C(X)_{\infty, \alpha} \cap C(X)' = C(X)_{\infty, \alpha} = C((\beta\mathbb{N} \setminus \mathbb{N}) \times X),$$

and that the induced action on $(\beta\mathbb{N} \setminus \mathbb{N}) \times X$ is trivial on $\beta\mathbb{N} \setminus \mathbb{N}$ and the G -action on X . The existence of a completely positive contractive order zero map $\varphi: C(G) \rightarrow C((\beta\mathbb{N} \setminus \mathbb{N}) \times X)$ is easily seen to be equivalent to the existence of an open set U in $(\beta\mathbb{N} \setminus \mathbb{N}) \times X$ and a continuous function $\phi: U \rightarrow G \times (0, 1]$. With this in mind, it is easy to see that φ is equivariant if and only if U is G -invariant and ϕ is equivariant. The rest of the proof is straightforward, and is omitted. \square

Theorem 4.2. Let G be a compact Lie group and let X be a compact Hausdorff space. Let G act on X and denote by $\alpha: G \rightarrow \text{Aut}(C(X))$ the induced action of G on $C(X)$.

- (1) If α has finite Rokhlin dimension, then the action of G on X is free.
- (2) If the action of G on X is free, then α has finite Rokhlin dimension. In fact, there are a non-negative integer d and equivariant completely positive contractive order zero maps

$$\varphi_0, \dots, \varphi_d: C(G) \rightarrow C(X)$$

such that $\sum_{j=0}^d \varphi_j(1) = 1$. Moreover, if $\dim(X) < \infty$, we have

$$\dim_{\text{Rok}}(\alpha) \leq \dim(X) - \dim(G).$$

We point out that the conclusion in part (2) above really is stronger than α having finite Rokhlin dimension, since one can choose the maps to land in $C(X)$ rather than in its (central) sequence algebra $C(X)_{\infty, \alpha} = C(X)_{\infty, \alpha} \cap C(X)'$.

Proof. Part (1). Assume that there exist $g \in G$ and $x \in X$ with $g \cdot x = x$. Choose an open cover U_0, \dots, U_n of $(\beta\mathbb{N} \setminus \mathbb{N}) \times X$ consisting of G -invariant open sets, and continuous equivariant functions $\phi_j: U_j \rightarrow G \times (0, 1]$ as in Lemma 4.1. Fix $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ and choose $j \in \{0, \dots, n\}$ such that $(\omega, x) \in U_j$. Write $\phi_j: U_j \rightarrow G \times (0, 1]$ as $\phi = (\phi^{(1)}, \phi^{(2)})$. Note that $\phi^{(1)}: U_j \rightarrow G$ is equivariant, where the action of G on itself is given by left translation (and in particular, it is free). We have

$$\left(\phi_j^{(1)}(\omega, x), \phi_j^{(2)}(\omega, x) \right) = \phi_j(\omega, x) = \phi_j(\omega, g \cdot x) = \left(g\phi_j^{(1)}(\omega, x), \phi_j^{(2)}(\omega, x) \right),$$

which implies that $\phi_j^{(1)}(\omega, x) = g\phi_j^{(1)}(\omega, x)$ and hence $g = 1$. The action of G on X is therefore free.

Part (2). The proof is almost identical to that of Theorem 3.10, using Theorem 1.1 in [29] in place of part (1) of Theorem 2 in [19]. (Since G is a Lie group, we do not need X to be finite dimensional for the quotient map $X \rightarrow X/G$ to be a principal G -bundle.) When X is not necessarily finite-dimensional, we simply take d to be the cardinality of some open cover \mathcal{U} consisting of open subsets of X/G over which the fiber bundle $X \rightarrow X/G$ is trivial. When $\dim(X) < \infty$, we have $\dim(X/G) = \dim(X) - \dim(G)$, so we can again use Proposition 1.5 in [22] to refine \mathcal{U} to a $(\dim(X) - \dim(G))$ -decomposable open cover of X/G , and proceed as in the proof of Theorem 3.10. We omit the details. \square

Remark 4.3. It follows from the dimension estimate in part (2) of the above theorem that whenever a compact Lie group G acts freely on a compact Hausdorff space X of the same dimension as G , then the induced action of G on $C(X)$

has the Rokhlin property. In fact, in this case it follows that X is equivariantly homeomorphic to $G \times (X/G)$, where the G -action on G is left translation and the action on X/G is trivial. Indeed, $\dim(X/G) = \dim(X) - \dim(G)$, so X/G is zero-dimensional. If $\pi: X \rightarrow X/G$ denotes the canonical quotient map, then by Theorem 8 in [25], there exists a continuous map $\lambda: X/G \rightarrow X$ such that $\pi \circ \lambda = \text{id}_{X/G}$. One easily checks that the map $X \rightarrow G \times (X/G)$ given by $x \mapsto (\lambda(\pi(x)), \pi(x))$ for $x \in X$, is a homeomorphism. It is also readily verified that it is equivariant, thus proving the claim.

Theorem 4.5 below leads to a useful criterion to determine when a given action of a compact group has finite Rokhlin dimension with commuting towers, although it is less useful if one is interested in the actual value of the Rokhlin dimension. For most applications, however, having the exact value is not as important as knowing that it is finite. In particular, it will follow from said theorem that for a compact Lie group, the X -Rokhlin property for a finite dimensional compact Hausdorff space X (as defined in Definition 1.5 in [13]), is equivalent to finite Rokhlin dimension with commuting towers in our sense.

We first need a lemma about universal C^* -algebras generated by the images of completely positive contractive order zero maps. We present the non-commuting tower version, as well as its commutative counterpart, for use in a later result. In the case of a finite group action with finite Rokhlin dimension with commuting towers, the result below was first obtained by Ilan Hirshberg, and its proof is contained in the proof of Lemma 1.6 in [13].

Lemma 4.4. Let G be a compact group and let d be a non-negative integer.

- (1) There exist a unital C^* -algebra C and an action $\gamma: G \rightarrow \text{Aut}(C)$ of G on C with the following universal property. Let B be a unital C^* -algebra, let $\beta: G \rightarrow \text{Aut}(B)$ be an action of G on B , and let $\varphi_0, \dots, \varphi_d: A \rightarrow B$ be equivariant completely positive contractive order zero maps such that $\varphi_0(1) + \dots + \varphi_d(1) = 1$. Then there exists a unital equivariant homomorphism $\varphi: C \rightarrow B$.
- (2) There exists a compact metrizable free G -space X with the following universal property. Let B be a unital C^* -algebra, let $\beta: G \rightarrow \text{Aut}(B)$ be an action of G on B , and let $\varphi_0, \dots, \varphi_d: A \rightarrow B$ be equivariant completely positive contractive order zero maps with commuting ranges such that $\varphi_0(1) + \dots + \varphi_d(1) = 1$. Then there exists a unital equivariant homomorphism $\varphi: C(X) \rightarrow B$.

Moreover, the space X in part (2) satisfies

$$\dim(X) \leq (d+1)(\dim(G)+1) - 1.$$

Proof. Part (1). Set

$$D = \ast_{j=0}^d C_0((0,1] \times G),$$

and let $\delta: G \rightarrow \text{Aut}(D)$ be the action obtained by letting G act on each of the free factors diagonally, with trivial action on $(0,1]$ and translation on G . Denote by I the ideal in D generated by

$$\left\{ \left(\sum_{j=0}^d \text{id}_{(0,1]} \ast 1_{C(G)} \right) c - c : c \in D \right\}.$$

Then I is δ -invariant, and hence there is an induced action γ of G on the unital quotient $C = D/I$. It is clear that the C^* -algebra C and the action γ are as desired.

Part (2). Set

$$D = \bigotimes_{j=0}^d C_0((0, 1] \times G),$$

and let $\delta: G \rightarrow \text{Aut}(D)$ be the action obtained by letting G act on each of the tensor factors diagonally. Then D is a commutative C^* -algebra and the action on its maximal ideal space induced by δ is free. Denote by I the ideal in D generated by

$$\left\{ \left(\sum_{j=0}^d \text{id}_{(0,1]} \otimes 1_{C(G)} \right) c - c : c \in D \right\}.$$

Then I is δ -invariant, and hence there is an induced action γ of G on the unital quotient $C = D/I$. Set $X = \text{Max}(C)$, which is a compact metrizable space. The action on X induced by γ is free, being the restriction of a free action to an invariant closed subset. It is clear that X is the desired free G -space.

The dimension estimate for X follows from the fact that it is a closed subset of $\bigotimes_{j=0}^d (0, 1] \times G$. \square

Theorem 4.5. Let G be a compact Lie group, let A be a unital C^* -algebra, and let $\alpha: G \rightarrow \text{Aut}(A)$ be a continuous action. Then α has finite Rokhlin dimension with commuting towers if and only if there exist a finite dimensional compact free G -space X and an equivariant unital embedding

$$\varphi: C(X) \rightarrow A_{\infty, \alpha} \cap A'.$$

Moreover, we have the following relations between the dimension of X and the Rokhlin dimension of α :

$$\begin{aligned} \dim(X) &\leq (\dim_{\text{Rok}}^c(\alpha) + 1)(\dim(G) + 1) - 1 \\ \dim_{\text{Rok}}^c(\alpha) &\leq \dim(X) - \dim(G) \end{aligned}$$

Proof. We begin by showing the “only if” implication. Let $d = \dim_{\text{Rok}}^c(\alpha)$, and denote by Y the compact metrizable free G -space obtained as in the conclusion of part (2) in Lemma 4.4. By universality of Y , there is a unital equivariant homomorphism

$$C(Y) \rightarrow A_{\infty, \alpha} \cap A'.$$

The kernel of this homomorphism is a G -invariant ideal of $C(Y)$ which has the form $C_0(U)$ for some G -invariant open subset U of Y . Set $X = Y \setminus U$ and denote by $\varphi: C(X) \rightarrow A_{\infty, \alpha} \cap A'$ the induced homomorphism. Then the G -action on X is free and φ is unital, equivariant and injective. Finally, we have

$$\dim(X) \leq \dim(Y) \leq (d + 1)(\dim(G) + 1) - 1,$$

and since G is a compact Lie group, it follows that X is finite dimensional.

We now show the “if” implication. Set $d = \dim(X) - \dim(G) + 1$ and choose completely positive contractive order zero maps

$$\varphi_0, \dots, \varphi_d: C(G) \rightarrow C(X)$$

as in the conclusion of part (2) of Theorem 4.2. It is immediate to show that the completely positive contractive order zero maps

$$\varphi \circ \varphi_0, \dots, \varphi \circ \varphi_d: C(G) \rightarrow A_{\infty, \alpha} \cap A'$$

satisfy the conditions in the definition of finite Rokhlin dimension with commuting towers for α . This finishes the proof. \square

In particular, it follows from Theorem 4.5 that for a compact Lie group, the X -Rokhlin property for a compact Hausdorff space X (as defined in Definition 1.5 of [13]), is equivalent to finite Rokhlin dimension with commuting towers.

Corollary 4.6. Let G be a compact Lie group, let A be a unital C^* -algebra, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action with $\dim_{\text{Rok}}^c(\alpha) < \infty$. Then α has discrete K -theory, this is, there is n in \mathbb{N} such that $I_G^n \cdot K_*^G(A) = 0$.

Proof. Corollary 4.3 in [13] asserts that compact Lie group actions with the X -Rokhlin property have discrete K -theory. The result follows from the fact that finite Rokhlin dimension with commuting towers implies the X -Rokhlin property by Theorem 4.5. \square

It is proved in [9] that a compact Lie group G acts on a unital C^* -algebra A with the Rokhlin property, then already I_G annihilates $K_*^G(A)$. It may therefore be tempting to conjecture that in the context of Corollary 4.6 above, one has

$$\dim_{\text{Rok}}^c(\alpha) + 1 = \min \{n: I_G^n \cdot K_*^G(A) = 0\},$$

or at least that the right-hand side determines $\dim_{\text{Rok}}^c(\alpha)$. This is unfortunately not in general the case, even for finite group actions on Kirchberg algebras that satisfy the UCT, as the following example shows.

Example 4.7. Let B and $\beta: \mathbb{T} \rightarrow \text{Aut}(B)$ be the C^* -algebra and the circle action from Example 3.13. As mentioned there, β has the Rokhlin property and for all m in \mathbb{N} , its restriction $\beta|_m$ to $\mathbb{Z}_m \subseteq \mathbb{T}$ does not have the Rokhlin property. Fix m in \mathbb{N} . It follows from Theorem 3.10 that $\dim_{\text{Rok}}^c(\beta|_m) = 1$. Moreover, by Lemma 4.15 in [5] and part (1) of Proposition 4.22 in [5], the dual action $\widehat{\beta}|_n: \mathbb{Z}_m \rightarrow \text{Aut}(B \rtimes \mathbb{Z}_m)$ is approximately inner. In particular $1 - K_*(\widehat{\beta}|_{n_1}) = 0$ and thus $I_{\mathbb{Z}_m} \cdot K_*^{\mathbb{Z}_m}(B) = 0$. If $\min \{n: I_{\mathbb{Z}_m}^n \cdot K_*^{\mathbb{Z}_m}(B) = 0\}$ determined the Rokhlin dimension of $\beta|_m$, we should have $\dim_{\text{Rok}}^c(\beta|_m) = 0$, and this would be a contradiction.

The phenomenon exhibited above can also be encountered among free actions on spaces, as the action of \mathbb{Z}_2 on S^1 by rotation shows. Finally, we mention that it is nevertheless conceivable that one has

$$\min \{n: I_G^n \cdot K_*^G(A) = 0\} \leq \dim_{\text{Rok}}^c(\alpha) + 1,$$

but we have not explored this direction any further.

Having discrete K -theory is special to actions with finite Rokhlin dimension *with commuting towers*, as the following example, which was constructed by Izumi in a different context, shows.

Let p be a projection in \mathcal{O}_∞ whose class in $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$ is 0. We recall that the *standard Cuntz algebra* \mathcal{O}_∞^{st} is defined to be the corner $p\mathcal{O}_\infty p$. It follows from Kirchberg-Phillips classification of Kirchberg algebras in the UCT class (see [20])

and [27]), that \mathcal{O}_∞^{st} is the unique unital Kirchberg algebra satisfying the UCT with K -theory given by

$$(K_0(\mathcal{O}_\infty^{st}), [1_{\mathcal{O}_\infty^{st}}], K_1(\mathcal{O}_\infty^{st})) \cong (\mathbb{Z}, 0, \{0\}).$$

Since the unit of \mathcal{O}_∞^{st} is trivial in K_0 , there is a unital homomorphism $\mathcal{O}_2 \rightarrow \mathcal{O}_\infty^{st}$. Hence there is an approximately central embedding of \mathcal{O}_2 into $\bigotimes_{n \in \mathbb{N}} \mathcal{O}_\infty^{st}$, so it follows from Theorem 3.8 in [21] that $\bigotimes_{n \in \mathbb{N}} \mathcal{O}_\infty^{st} \cong \mathcal{O}_2$.

Example 4.8. (See the example on page 262 of [16].) Let p be a projection in \mathcal{O}_∞ whose class in $K_0(\mathcal{O}_\infty)$ is 0, and set $u = 2p - 1$. Then u is a unitary of \mathcal{O}_∞ which leaves the corner $p\mathcal{O}_\infty p \cong \mathcal{O}_\infty^{st}$ invariant. Since $\bigotimes_{n \in \mathbb{N}} \mathcal{O}_\infty^{st}$ is isomorphic to \mathcal{O}_2 , if we let α be the infinite tensor product automorphism $\alpha = \bigotimes_{n \in \mathbb{N}} \text{Ad}(u)$ of \mathcal{O}_2 , then α determines a \mathbb{Z}_2 action on \mathcal{O}_2 .

It is shown in Remark 2.5 in [2] that α has Rokhlin dimension 1 with non-commuting towers. We claim that $\dim_{\text{Rok}}^c(\alpha) = \infty$, this is, that α has infinite Rokhlin dimension with commuting towers. We show that α does not have discrete K -theory, and the result will then follow from Corollary 4.6.

It is shown in [16] that $\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2$ is isomorphic to a direct limit of $B_n = \mathcal{O}_\infty^{st} \oplus \mathcal{O}_\infty^{st}$ with connecting maps that on K_0 are stationary and given by the matrix

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Moreover, the dual action $\hat{\alpha}: \widehat{\mathbb{Z}_2} \cong \mathbb{Z}_2 \rightarrow \text{Aut}(\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2)$ is the direct limit of the actions $\gamma_n: \mathbb{Z}_2 \rightarrow \text{Aut}(B_n)$ given by $\gamma_n(a, b) = (b, a)$ for all $(a, b) \in B_n = \mathcal{O}_\infty^{st} \oplus \mathcal{O}_\infty^{st}$. It follows that $\hat{\alpha}$ is multiplication by -1 on $K_0(\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2)$ (it is given by exchanging the columns in the above matrix). It is shown in Lemma 4.7 in [16] that $K_0(\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2) \cong \mathbb{Z}[\frac{1}{2}]$. Given n in \mathbb{N} , we have

$$I_{\mathbb{Z}_2}^n \cdot K_0^{\mathbb{Z}_2}(\mathcal{O}_2) \cong \text{Im}(\text{id}_{K_0(\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2)} - K_0(\hat{\alpha}))^n.$$

Now, $\text{id}_{K_0(\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2)} - K_0(\hat{\alpha})$ is multiplication by 2 on $\mathbb{Z}[\frac{1}{2}]$, so

$$(\text{id}_{K_0(\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2)} - K_0(\hat{\alpha}))^n$$

is an isomorphism for all n in \mathbb{N} , and in particular α does not have discrete K -theory. This proves the claim.

It follows from the example above that the notions of Rokhlin dimension with and without commuting towers do not in general agree. Even more, having *finite* Rokhlin dimension without commuting towers is really weaker than having *finite* Rokhlin dimension with commuting towers. Such phenomenon can happen even if the K -theoretic obstructions found in [13] vanish. This answers a question that was implicitly left open in [15], at least in the finite (and compact) group case. We do not know whether there are similar examples for automorphisms.

Finite Rokhlin dimension with commuting towers is not in general equivalent to having discrete K -theory, since the trivial action on \mathcal{O}_2 clearly does not have finite Rokhlin dimension but has discrete K -theory for trivial reasons. The following example, which was originally constructed by Phillips with a different purpose, shows that absence of K -theory is not the only thing that can go wrong.

Example 4.9. We recall the construction in Example 9.3.9 in [28] of an AF-action of \mathbb{Z}_4 on the CAR algebra M_{2^∞} whose restriction to \mathbb{Z}_2 is not K -free, and show that it has other interesting properties.

For n in \mathbb{N} , let $A_n = M_{2^n}(\mathbb{C} \oplus \mathbb{C})$ and set $u_n = \bigotimes_{k=1}^n \text{diag}(1, -1)$, which is a unitary in M_{2^n} (not in A_n). Define connecting maps $\iota_n: A_n \rightarrow A_{n+1}$ by

$$\iota_n(a, b) = \left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & u_n a u_n^* \end{pmatrix} \right)$$

for $(a, b) \in A_n$. Define an automorphism $\alpha^{(n)}$ of A_n by $\alpha^{(n)}(a, b) = (u_n b u_n^*, a)$ for $(a, b) \in A_n$. Since u_n has order two, it is easy to see that $\alpha^{(n)}$ has order four, so it defines an action of \mathbb{Z}_4 on A_n . It is also readily checked that there is a direct limit action $\alpha = \varinjlim \alpha^{(n)}$ of \mathbb{Z}_4 on $A = \varinjlim A_n$. Finally, the direct limit algebra A is easily seen to be isomorphic to the CAR algebra M_{2^∞} by classification.

As proved in [28], with $p = (1, 0) \in \mathbb{C} \oplus \mathbb{C} \subseteq A$, it is easy to show that α^2 is the action of conjugation by the unitary $2p - 1$, so $\alpha|_{\mathbb{Z}_2}$ is in fact inner. In particular, α is not pointwise outer so it does not have finite Rokhlin dimension, with or without commuting towers, by Theorem 4.14 below.

The crossed product $A \rtimes_\alpha \mathbb{Z}_4$ is the direct limit of the inductive system

$$A_1 \otimes C^*(\mathbb{Z}_4) \rightarrow A_2 \otimes C^*(\mathbb{Z}_4) \rightarrow \cdots \rightarrow A \rtimes_\alpha \mathbb{Z}_4.$$

The computation of the connecting maps is routine, and yields an isomorphism $A \rtimes_\alpha \mathbb{Z}_4 \cong M_{2^\infty}$, which is best seen using Bratteli diagrams. (Alternatively, one can compute the equivariant K -theory of A , as is done in [28].) To show that α has discrete K -theory, it suffices to observe that the dual action acts via approximately inner automorphisms, since every automorphism of a UHF-algebra is approximately inner. In particular,

$$I_{\mathbb{Z}_4} \cdot K_*^{\mathbb{Z}_4}(A) \cong \text{Im}(1 - K_*(\hat{\alpha}_1)) = 0,$$

as desired.

We turn to the comparison with locally discrete K -theory and total K -freeness.

Definition 4.10. (See Definitions 4.1.1, 4.2.1 and 4.2.4 of [28].) Let G be a compact group, let A be a unital C^* -algebra and let $\alpha: G \rightarrow \text{Aut}(A)$ be a continuous action.

- (1) We say that α has *locally discrete K -theory* if for every prime ideal P of $R(G)$ not containing the augmentation ideal I_G , the localization $K_*^G(A)_P$ is zero.
- (2) We say that α is *K -free* if for every invariant ideal I of A , the induced action $\alpha|_I: G \rightarrow \text{Aut}(I)$ has locally discrete K -theory.
- (3) We say that α is *totally K -free* if for every closed subgroup H of G , the restriction $\alpha|_H$ is K -free.

Corollary 4.11. Let G be a compact Lie group, let A be a unital C^* -algebra and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action with finite Rokhlin dimension with commuting towers. Then α has locally discrete K -theory.

Proof. The action α has discrete K -theory by Corollary 4.6. It then follows from the equivalence between (1) and (2) in Proposition 4.1.3 of [28] that α has locally discrete K -theory. \square

Corollary 4.12. Let G be a compact Lie group, let A be a unital C^* -algebra and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action with finite Rokhlin dimension with commuting towers. Then α is totally K -free.

Proof. Let H be a closed subgroup of G and let I be an H -invariant ideal of A . Since the restriction of α to H has finite Rokhlin dimension with commuting towers by Theorem 3.10, we may assume that $H = G$, so that I is G -invariant. We have to show that the induced action of G on I has locally discrete K -theory.

Since the induced action of G on A/I has finite Rokhlin dimension with commuting towers by part (2) of Theorem 3.8, it follows from the corollary above that it has locally discrete K -theory. In particular, the extension

$$0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$$

is G -equivariant, and the actions on A and A/I have locally discrete K -theory. The result now follows from Lemma 4.1.4 of [28]. \square

Even total K -freeness is not equivalent to finite Rokhlin dimension.

Example 4.13. Let α be the trivial action of \mathbb{Z}_2 on \mathcal{O}_2 . Then α is readily seen to be totally K -free, but it clearly does not have finite Rokhlin dimension, with or without commuting towers, by Theorem 4.14 below.

Recall that an action of a locally compact group G on a C^* -algebra A is said to be *pointwise outer* (and sometimes just *outer*), if for every $g \in G \setminus \{1\}$, the automorphism α_g of A is not inner.

Theorem 4.14. Let A be a unital C^* -algebra, let G be a compact Lie group, and let $\alpha: G \rightarrow \text{Aut}(A)$ be a continuous action. If $\dim_{\text{Rok}}(\alpha) < \infty$, then α is pointwise outer.

We point out that we do not assume that α has finite Rokhlin dimension with commuting towers, unlike in most other results in this section.

Proof. Let $g \in G \setminus \{1\}$ and assume that α_{g_0} is inner, say $\alpha_{g_0} = \text{Ad}(u)$ for some $u \in \mathcal{U}(A)$. Let C , let $\gamma: G \rightarrow \text{Aut}(C)$ and let

$$\varphi: C \rightarrow A_{\infty, \alpha} \cap A'$$

be the unital C^* -algebra, the action of G on C , and the unital equivariant homomorphism obtained as in the conclusion of part (1) of Lemma 4.4.

We claim that there exists a positive element a in C with the following properties:

- The elements a and $\gamma_g(a)$ are orthogonal.
- $\|\varphi(a)\| = \|\varphi(\gamma_g(a))\| = \|\varphi(a) - \varphi(\gamma_g(a))\| = 1$.

Set $d = \dim_{\text{Rok}}(\alpha)$. Recall from the proof of Lemma 4.4 that C is the quotient of the C^* -algebra

$$D = \ast_{j=0}^d C_0((0, 1] \times G)$$

by the ideal I generated by

$$\left\{ \left(\sum_{j=0}^d \text{id}_{(0,1]} \ast 1_{C(G)} \right) c - c : c \in D \right\}.$$

Denote by $\pi: D \rightarrow C$ the quotient map. Choose a positive function f in $C(G)$ such that the supports of $\text{Lt}_g(f)$ and f are disjoint. Set $b = \text{id}_{(0,1]} \otimes f \in C_0((0, 1]) \otimes C(G)$,

and regard it as an element in D via the embedding of $C_0((0, 1]) \otimes C(G)$ as the first free factor. We claim that $\pi(b) \neq 0$. Indeed, if $\pi(b) = 0$, then $\pi(\delta_h(b)) = 0$ for all h in G . Since the action of translation of G on itself is transitive, we conclude that the first free factor of D is contained in the kernel of π . Now, this contradicts the fact that $d = \dim_{\text{Rok}}(\alpha)$, since it shows that the definition of finite Rokhlin dimension for α is satisfied with $d - 1$ order zero maps. This shows that $\pi(b) \neq 0$.

Upon renormalizing b , we may assume that $a = \pi(b)$ is positive and has norm 1. It is clear that a and $\gamma_g(a) = \pi(\delta_g(b))$ are orthogonal, and that $\gamma_g(a)$ is positive and has norm 1. Finally, it follows from orthogonality of a and $\gamma_g(a)$ that $\|\varphi(a) - \varphi(\gamma_g(a))\| = 1$. This proves the claim.

Let $\varepsilon = \frac{1}{3}$. Using Choi-Effros lifting theorem, find a completely positive contractive map $\psi: C \rightarrow A$ satisfying the following conditions:

- (1) $\|[\psi(a), u]\| < \varepsilon$;
- (2) $\|\psi(\gamma_g(a)) - \alpha_g(\psi(a))\| < \varepsilon$;
- (3) $\|\psi(a) - \psi(\gamma_g(a))\| - 1 < \varepsilon$.

We have

$$\begin{aligned} \frac{2}{3} &= 1 - \varepsilon < \|\psi(a) - \psi(\gamma_g(a))\| \leq \varepsilon + \|\psi(a) - \alpha_g(\psi(a))\| \\ &= \varepsilon + \|\psi(a) - u\psi(a)u^*\| \leq 2\varepsilon = \frac{2}{3}, \end{aligned}$$

which is a contradiction. This contradiction implies that α_g is not inner, thus showing that α is pointwise outer. \square

In the case of commuting towers, the converse to the theorem above fails quite drastically, and there are many examples of compact group actions that are pointwise outer and have infinite Rokhlin dimension with commuting towers. See Example 3.9, where it is shown that the gauge action on \mathcal{O}_2 has infinite Rokhlin dimension with commuting towers, and see Example 4.8 for an example where the acting group is \mathbb{Z}_2 . The second one has finite Rokhlin dimension with non-commuting towers (in fact, Rokhlin dimension 1). We do not know whether the Rokhlin dimension of the gauge action on \mathcal{O}_2 is finite. It is known, however, that all of its restrictions to finite subgroups of \mathbb{T} have Rokhlin dimension with non commuting towers equal to 1.

On the other hand, we do not know exactly how badly the converse to the theorem above fails in the case of non-commuting towers, although we know it does not hold in full generality.

Example 4.15. The action α of \mathbb{Z}_2 on S^1 given by conjugation has two fixed points, so it is not free, and hence $\dim_{\text{Rok}}(\alpha) = \infty$. On the other hand, α is certainly pointwise outer since it is not trivial.

4.1. A rigidity result. Using the fact that compact group actions with finite Rokhlin dimension with commuting towers are totally K -free by Corollary 4.12, we are able to obtain a certain K -theoretical obstruction for a unital C^* -algebra to admit such an action. As a consequence of this K -theoretical obstruction, we confirm a conjecture of Phillips.

Theorem 4.16. Let G be a compact Lie group, let A be a C^* -algebra, and let $\alpha: G \rightarrow \text{Aut}(A)$ be a continuous action. Assume that one and only one of either $K_0(A)$ or $K_1(A)$ vanishes. If α is totally K -free, then G is finite.

Proof. We will show the result assuming that $K_1(A) = 0$; the corresponding proof for the case $K_0(A) = 0$ is analogous.

By restricting to the connected component of the identity in G , and recalling that K -freeness passes to subgroups, we can assume that G is connected. Assume that G is not the trivial group. By further restricting to any copy of the circle inside a maximal torus, we may assume that $G = \mathbb{T}$. Having discrete K -theory, there exists $n \in \mathbb{N}$ such that $I_{\mathbb{T}}^n \cdot K_*^{\mathbb{T}}(A) = 0$. Equivalently,

$$\ker((\text{id}_{K_*(A \rtimes_{\alpha} \mathbb{T})} - \hat{\alpha}_*)^n) = K_*(A \rtimes_{\alpha} \mathbb{T}).$$

Using that $K_1(A) = 0$, it follows from the Pimsner-Voiculescu exact sequence associated to α (see Subsection 10.6 in [3]),

$$\begin{array}{ccccc} K_0(A \rtimes_{\alpha} \mathbb{T}) & \xrightarrow{1-K_0(\hat{\alpha})} & K_0(A \rtimes_{\alpha} \mathbb{T}) & \longrightarrow & K_0(A) \\ \uparrow & & & & \downarrow \\ K_1(A) & \longleftarrow & K_1(A \rtimes_{\alpha} \mathbb{T}) & \xleftarrow{1-K_1(\hat{\alpha})} & K_1(A \rtimes_{\alpha} \mathbb{T}), \end{array}$$

that the map $\text{id}_{K_0(A \rtimes_{\alpha} \mathbb{T})} - \hat{\alpha}_0$ is injective. This implies that $K_0(A \rtimes_{\alpha} \mathbb{T}) = 0$ and the remaining potentially non-zero terms in the Pimsner-Voiculescu exact sequence yield the short exact sequence

$$0 \rightarrow K_0(A) \rightarrow K_1(A \rtimes_{\alpha} \mathbb{T}) \rightarrow K_1(A \rtimes_{\alpha} \mathbb{T}) \rightarrow 0,$$

where the last map is $\text{id}_{K_1(A \rtimes_{\alpha} \mathbb{T})} - \hat{\alpha}_1$. Being surjective, every power of it is surjective, and hence the identity

$$\ker((\text{id}_{K_1(A \rtimes_{\alpha} \mathbb{T})} - \hat{\alpha}_1)^n) = K_1(A \rtimes_{\alpha} \mathbb{T})$$

forces $K_1(A \rtimes_{\alpha} \mathbb{T}) = 0$. In this case, it must be $K_0(A) = 0$ as well, which contradicts the fact that $K_0(A)$ is not zero. \square

Recall that an AF-action is an action on an AF-algebra obtained as a direct limit of actions on finite dimensional C^* -algebras. It was shown in [4] that not every action on an AF-algebra is an AF-action, even when the group is \mathbb{Z}_2 .

Remark 4.17. Conjecture 9.4.9 in [28] says that there does not exist a totally K -free AF-action of a non-trivial connected compact Lie group on an AF-algebra. Theorem 4.16 above confirms this conjecture of Phillips, for a much larger class of C^* -algebras, and without assuming that the action is specified by the way it is constructed.

Corollary 4.18. No non-finite compact Lie group admits an action with finite Rokhlin dimension with commuting towers on a C^* -algebra with exactly one vanishing K -group. In particular, there are no such actions on AF-algebras, AI-algebras, the Cuntz algebras \mathcal{O}_n for $n \in \{3, \dots, \infty\}$, or the Jiang-Su algebra \mathcal{Z} .

We make some comments about what happens for finite groups. Many AF-algebras (although not all of them) as well as all Cuntz algebras \mathcal{O}_n with $n < \infty$, admit finite group actions with finite Rokhlin dimension. In fact, they even admit actions of some finite groups with the Rokhlin property (although there are severe

restrictions on the cardinality of the group in each case). On the other hand, Theorem 4.7 in [13] asserts that \mathcal{O}_∞ and \mathcal{Z} do not admit *any* finite group action with finite Rokhlin dimension.

We specialize to AF-algebras now, since a little more can be said in this case. Recall that an action α of a locally compact group G on a unital C^* -algebra is said to be *inner* if there exists a continuous group homomorphism $u: G \rightarrow \mathcal{U}(A)$ such that $\alpha_g = \text{Ad}(u(g))$ for all $g \in G$.

Definition 4.19. An AF-action α of a locally compact group G on a unital C^* -algebra A is said to be *locally representable* if there exists a sequence $(A_n)_{n \in \mathbb{N}}$ of unital finite dimensional subalgebras of A such that

- $\bigcup_{n \in \mathbb{N}} A_n$ is dense in A
- $\alpha_g(A_n) \subseteq A_n$ for all $g \in G$ and all n in \mathbb{N} .
- $\alpha|_{A_n}$ is inner for all n in \mathbb{N} .

Product type actions on UHF-algebras are examples of locally representable actions. Such actions have been classified in terms of their equivariant K -theory by Handelman and Rossmann in [11].

Using Theorem 4.16 and a result from [28], we are able to describe all locally representable actions α of a compact Lie group G on an AF-algebra with $\dim_{\text{Rok}}^c(\alpha) < \infty$: the group G must be finite, and all such actions are conjugate to a specific model action μ^G , so in particular they all have the Rokhlin property. The model action $\mu^G: G \rightarrow \text{Aut}(M_{|G|^\infty})$ is the infinite tensor product of copies of the left regular representation. (We identify $M_{|G|}$ with $\mathcal{K}(\ell^2(G))$ in the usual way.) It is well known that μ^G (and any tensor product of it with any other action) has the Rokhlin property; see [16].

Corollary 4.20. Let G be a compact Lie group, let A be a unital AF-algebra, and let $\alpha: G \rightarrow \text{Aut}(A)$ be a locally representable AF-action. Then the following are equivalent:

- (1) α has the Rokhlin property;
- (2) α has finite Rokhlin dimension with commuting towers;
- (3) α is totally K -free;
- (4) α has discrete K -theory.

Moreover, if any of the above holds, then G must be finite and there is an equivariant isomorphism

$$(A, \alpha) \cong (A \otimes M_{|G|^\infty}, \text{id}_A \otimes \mu^G).$$

In particular, α absorbs μ^G tensorially.

Proof. The implications (1) \Rightarrow (2) \Rightarrow (3) are true in general. The equivalence between (3) and (4) follows from Theorem 9.2.4 of [28]. In particular, any of the conditions (1) through (4) implies that α is totally K -free, so G must be finite by Theorem 4.16. Now, the fact that the second condition implies the fifth in Theorem 9.2.4 of [28], which in turn follows from the classification results in [11], shows that (3) implies the existence of an equivariant isomorphism

$$(A, \alpha) \cong (A \otimes M_{|G|^\infty}, \text{id}_A \otimes \mu^G).$$

We conclude that α has the Rokhlin property, so (3) implies (1).

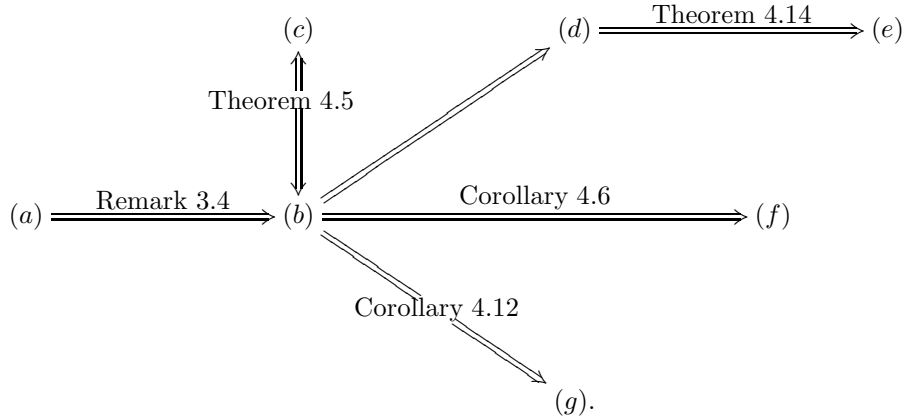
The final claim follows from the fact that μ^G absorbs itself tensorially. \square

We close this section by summarizing the known implications between the notions we have studied in this section.

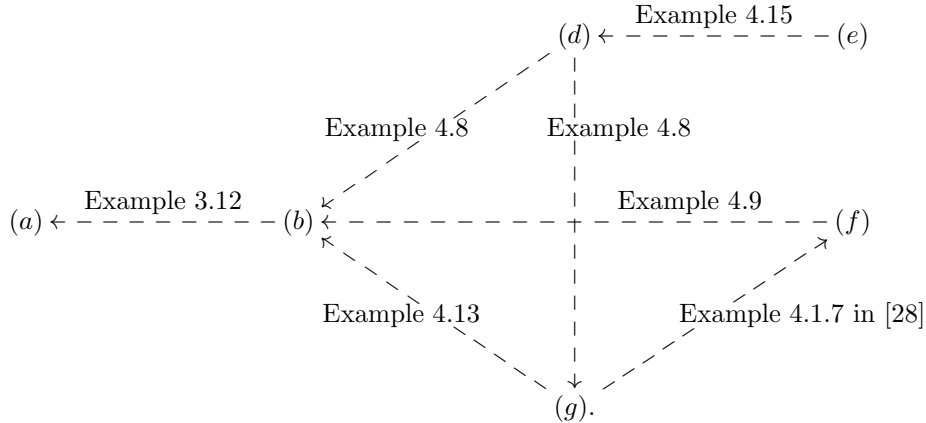
Theorem 4.21. Let G be a compact Lie group, let A be a unital C^* -algebra and let $\alpha: G \rightarrow \text{Aut}(A)$ be a continuous action. Consider the following conditions for the action α :

- (a) Rokhlin property.
- (b) Finite Rokhlin dimension with commuting towers, $\dim_{\text{Rok}}^c(\alpha) < \infty$.
- (c) X -Rokhlin property for some free G -space X .
- (d) Finite Rokhlin dimension, $\dim_{\text{Rok}}(\alpha) < \infty$.
- (e) Pointwise outerness.
- (f) Discrete K -theory.
- (g) Total K -freeness.

We have the following implications, where a theorem or corollary is referenced when it proves the implication in question:



None the above arrows can be reversed in full generality, and presumably there are no other implications between the stated conditions. In the diagram below, a dotted arrow means that the implication does not hold in general, and in each case a counterexample is referenced:



Finally, some of the arrows in the first diagram can be reversed in special situations:

- (1) If A is commutative, then conditions (b) , (c) , (d) , (f) and (g) are all equivalent to each other, and equivalent to freeness of the action on the maximal ideal space, by Theorem 4.5 and Atiyah-Segal completion theorem (see, for example, Theorem 1.1.1 in [28]). Condition (e) is not equivalent to the others by Example 4.15, and neither is condition (a) by Example 3.12.
- (2) If A is an AF-algebra and α is a locally representable AF-action (see Definition 4.19), then conditions (a) , (b) , (c) , (f) and (g) are equivalent by Corollary 4.20.
- (3) If A is a Kirchberg algebra and $G = \mathbb{Z}_2$ (and possibly also if G is any finite group), then (e) and (f) are equivalent by Theorem 2.3 in [2].

5. OUTLOOK AND OPEN PROBLEMS

In this last section, we give some indication of possible directions for future work, and raise some natural questions related to our findings.

Although some of our results, particularly in Section 4, assume that the acting group is a Lie group, this is probably not needed everywhere. Our first suggested problem is then:

Problem 5.1. Extend some of the results in this paper to actions of not necessarily finite-dimensional compact groups.

We point out that the assumption that G be a Lie group in Corollary 4.3 in [13] is necessary, since it relies on Atiyah-Segal completion Theorem (see [1], and see Theorem 1.1.1 in [28]), for which one needs the representation ring $R(G)$ to be finitely generated. We suspect that Corollary 4.6 is not true in general for arbitrary compact groups (it probably already fails for actions on compact spaces), but it may be the case that all compact group actions with finite Rokhlin dimension with commuting towers have *locally* discrete K -theory.

Somewhat related, we ask:

Question 5.2. Is finite Rokhlin dimension with commuting towers equivalent to the X -Rokhlin property for *arbitrary* compact groups?

Maybe one should start with the commutative case:

Question 5.3. Is finite Rokhlin dimension for actions on commutative C^* -algebras, equivalent to freeness of the induced action on the maximal ideal space for *arbitrary* compact groups?

For compact Lie groups, the answer is yes in both cases; see Theorem 4.5 and Theorem 4.2. It was crucial in the proofs of said theorems that free actions of compact Lie groups have local cross-sections. We suspect that the answer to Question 5.2 and Question 5.3 is ‘no’, and it is possible that the action of $G = \prod_{n \in \mathbb{N}} \mathbb{Z}_n$ on $X = \prod_{n \in \mathbb{N}} S^1$ by coordinate-wise rotation is a counterexample (to both questions). Such action has the X -Rokhlin property and is free, but the quotient map $X \rightarrow X/G$ is known not to have local cross-sections, since X is not locally homeomorphic to $X/G \times G$. We have not checked, however, whether this action has finite Rokhlin dimension.

The results in [2] show that for \mathbb{Z}_2 -actions on Kirchberg algebras, pointwise outerness implies Rokhlin dimension at most 1 with noncommuting towers. It is conceivable that a similar result holds for a larger class of finite groups, and presumably all of them.

Question 5.4. Is pointwise outerness equivalent to finite Rokhlin dimension (with noncommuting towers) for finite group actions on Kirchberg algebras?

If the question above has an affirmative answer, as the results in [2] suggest, one may try to prove the corresponding result for simple unital separable nuclear C^* -algebras with tracial rank zero, where presumably some additional assumptions will be needed.

Alternatively,

Problem 5.5. Find obstructions (not necessarily K -theoretical) for having an action of a finite (or compact) group with finite Rokhlin dimension with noncommuting towers.

The results in Section 4 of [5] suggest the following:

Conjecture 5.6. Let G be a compact Lie group and let $\alpha: G \rightarrow \text{Aut}(\mathcal{O}_2)$ be an action with finite Rokhlin dimension with commuting towers. Then α has the Rokhlin property.

Example 4.8 shows that the corresponding statement for noncommuting towers is in general false.

Based on Corollary 4.6 and the comments and examples after it, we ask:

Question 5.7. If $\alpha: G \rightarrow \text{Aut}(A)$ is an action of a compact Lie group on a unital C^* -algebra A and $\dim_{\text{Rok}}^c(\alpha) < \infty$, does one have

$$\min \{n: I_G^n \cdot K_*^G(A) = 0\} \leq \dim_{\text{Rok}}^c(\alpha) + 1,$$

or any other relationship between these quantities?

Example 4.7 shows that one cannot in general expect equality to hold.

Finally, the following problem is likely to be challenging:

Problem 5.8. Can actions with finite Rokhlin dimension with commuting towers on unital Kirchberg algebras satisfying the UCT be classified, in a way similar to what was done in [17] for finite group actions with the Rokhlin property, or in [6] and [7] for circle actions with the Rokhlin property?

Some of these questions will be addressed in [10].

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